

HYPERFINE STRUCTURE OF S-STATES IN MUONIC HELIUM ION

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Corrections of orders α^5 and α^6 are calculated in the hyperfine splittings of $1S$ and $2S$ - energy levels in the ion of muonic helium. The electron vacuum polarization effects, the nuclear structure corrections and recoil corrections are taken into account. The obtained numerical values of the hyperfine splittings -1334.56 meV ($1S$ state), -166.62 meV ($2S$ state) can be considered as a reliable estimate for the comparison with the future experimental data. The hyperfine splitting interval $\Delta_{12} = (8\Delta E^{hfs}(2S) - \Delta E^{hfs}(1S)) = 1.64$ meV can be used for the check of quantum electrodynamics.

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I. INTRODUCTION

The ion of muonic helium $(\mu_2^3He)^+$ is the bound state of the negative muon and helion (3_2He). The lifetime of this simple atom is determined by the muon decay in a time $\tau_\mu = 2.19703(4) \times 10^{-6}$ s. An increase of the lepton mass in muonic hydrogenic atoms as compared with electronic hydrogenic atoms ($m_\mu/m_e = 206.7682838(54)$ [1]) leads to the growth of three effects in the energy spectrum: the electron vacuum polarization, the nuclear structure and polarizability, the nuclear recoil effect. The first effect is important for the spectrum of muonic helium ion $(\mu_2^3He)^+$ because the ratio of the Compton wavelength of the electron to the Bohr radius of the atom $\mu Z\alpha/m_e \approx 1.45415$ is sufficiently close to the unity. The second effect of the nuclear structure is of the utmost importance because the muonic wave function has a significant overlap with the nucleus. Finally, the increase of the recoil effects is determined by the ratio of the muon and helion masses $m_\mu/m({}^3_2He) \approx 0.0376$ [1]. Despite the fact that this ratio is small, it is many times larger than the fine structure constant α ($\alpha^{-1} = 137.03599911(46)$ [1]). Moreover, some recoil effects contain the peculiar logarithm of the muon to helion mass ratio $\ln(m({}^3_2He)/m_\mu) \approx 3.28$ what leads to the numerical growth of the contribution.

High sensitivity of the bound muon characteristics to the distributions of the nuclear charge and magnetic moment in light muonic atoms (muonic hydrogen, ions of muonic helium) can be used for more precise determination of the charge radii of the proton, deuteron, helion and α - particle [2, 3, 4]. Moreover, the measurement of the hyperfine structure of light muonic atoms allows to obtain more precise values of the Zemach radii and to improve the accuracy of the theoretical calculations of the hyperfine splittings in the corresponding

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electronic atoms.

Theoretical investigations of the energy levels in light muonic atoms (the nuclear charge $Z = 1, 2$) were carried out many years ago in Refs. [5, 6, 7, 8, 9] on the basis of the Dirac equation. In the calculations of the energy intervals $(2P_{3/2} - 2S_{1/2})$, $(2P_{1/2} - 2S_{1/2})$ in the ions of muonic helium (${}^4_2\text{He}$, ${}^3_2\text{He}$) the different type corrections were considered with the precision 0.01 meV . The energy transitions $(2S - 2P)$ in the ion of muonic helium ($\mu_2^3\text{He}$) were calculated in [10, 11] with regard to the hyperfine structure on the basis of the Dirac equation with the accuracy 0.1 meV .

Although muonic atoms ($\mu_1^1\text{H}$), ($\mu_1^2\text{H}$), ($\mu_2^3\text{He}$)⁺, ($\mu_2^4\text{He}$)⁺ could be used for yet another check of quantum electrodynamics, the experimental study of the energy levels of these atoms lags behind the theory. It is appropriate at this point to recall the experiment for the Lamb shift $(2P - 2S)$ measurement in muonic hydrogen which is carried out many years at PSI (Paul Sherrer Institute) [12, 13] but didn't give while the positive result with the necessary accuracy 30 ppm. The only experiment with the successful end was performed on the muon beam at CERN [14, 15] with muonic helium ($\mu_2^4\text{He}$)⁺. Two resonance transitions with the wavelengths 811.68(15) nm and 897.6(3) nm corresponding to the fine structure intervals $(2P_{3/2} - 2S_{1/2})$ and $(2P_{1/2} - 2S_{1/2})$ were observed. In a later experiment [16] the resonance transition in the range $811.4 \leq \lambda \leq 812.0 \text{ nm}$ was not observed. So, at present new measurement of the Lamb shift and the hyperfine structure in the atoms ($\mu_2^3\text{He}$)⁺, ($\mu_2^4\text{He}$)⁺ is required.

There is a need to remark that in the last years the accuracy of the theoretical investigations of the energy spectra of simple atoms was increased essentially [3]. New QED corrections of order α^6 and α^7 in the energy spectra of muonium, positronium, hydrogen atom and ions of electronic helium were calculated [4]. For a number of hydrogenic atoms (hydrogen atom, ions of helium) the comparison of results of the theoretical investigations in QED with the experiment is difficult because the theoretical error in the calculation of the nuclear structure and polarizability contributions both for the Lamb shift and hyperfine structure remains very large and exceeds considerably the experimental errors. The progress in this field can be achieved due to new experimental investigations of the structure and polarizability of the proton and other nucleus and the use of light muonic atoms.

It is important to keep in mind that all contributions to the energy spectra of light muonic atoms can be divided into two groups. The corrections of the first group were obtained in the analytical form in the study of the energy levels of muonium, positronium and hydrogen atom. The second group includes numerous corrections of the electron vacuum polarization, the nuclear structure, recoil effects which are specific for each muonic atom. The aim of this work consists in the analytical and numerical calculation of corrections of orders α^5 and α^6 in the hyperfine structure of S -states in the muonic helium ion ($\mu_2^3\text{He}$)⁺ on the basis of the quasipotential method in quantum electrodynamics [17, 18]. We consider such effects of the electron vacuum polarization, recoil and nuclear structure corrections which have the crucial importance to attain the high accuracy of the calculation. Numerical values of corrections are obtained with the precision 0.001 meV . So, the purpose of our investigation consists in the improvement of the earlier performed calculations [6, 7, 8, 9] of the hyperfine splitting in the ion of muonic helium and derivation of the reliable estimates in the hyperfine structure of $1S$ and $2S$ -states which could be used in conducting a corresponding experiments. Modern numerical values of fundamental physical constants are taken from the paper [1]: the electron mass $m_e = 0.510998918(44) \cdot 10^{-3} \text{ GeV}$, the muon mass $m_\mu = 0.1056583692(94) \text{ GeV}$, the fine structure constant $\alpha^{-1} = 137.03599911(46)$, the helium mass

$m(^3_2He) = 2.80839142(24)$ GeV, the helium magnetic moment $\mu_h = -2.127497723(25)$ in the nuclear magnetons, the muon anomalous magnetic moment $a_\mu = 1.16591981(62) \cdot 10^{-3}$.

II. EFFECTS OF ONE-LOOP AND TWO-LOOP VACUUM POLARIZATION IN THE ONE-PHOTON INTERACTION

Our approach to the investigation of the hyperfine structure (HFS) in the muonic helium ion is based on the quasipotential method in quantum electrodynamics [19, 20, 21], where the two-particle bound state is described by the Schroedinger equation. The basic contribution to the interaction operator of the muon and helion for S -states is determined by the Breit Hamiltonian [22]:

$$H_B = H_0 + \Delta V_B^{fs} + \Delta V_B^{hfs}, \quad H_0 = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r}, \quad (1)$$

$$\Delta V_B^{fs} = -\frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta(\mathbf{r}) - \frac{Z\alpha}{2m_1 m_2 r} \left(\mathbf{p}^2 + \frac{\mathbf{r}(\mathbf{r}\mathbf{p}\mathbf{p})}{r^2} \right), \quad (2)$$

$$\Delta V_B^{hfs} = \frac{8\pi\alpha\mu_h}{3m_1 m_p} \frac{\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2}{4} \delta(\mathbf{r}), \quad (3)$$

where m_1 , m_2 are the muon and helion masses, m_p is the proton mass, μ_h is the helion magnetic moment. The potential of spin-spin interaction (3) gives the main contribution to the energy of hyperfine splitting of S -states (the Fermi energy). Averaging (3) over the Coulomb wave functions of $1S$ and $2S$ states

$$\psi_{100}(r) = \frac{W^{3/2}}{\sqrt{\pi}} e^{-Wr}, \quad W = \mu Z\alpha, \quad (4)$$

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-Wr/2} \left(1 - \frac{Wr}{2} \right), \quad (5)$$

we obtain the following result (the difference of the triplet and singlet states):

$$\Delta E_F^{hfs}(nS) = \frac{8\mu^3 Z^3 \alpha^4 \mu_h}{3m_1 m_p n^3} = \begin{cases} 1S : & -1370.725 \text{ meV} \\ 2S : & -171.341 \text{ meV} \end{cases}, \quad (6)$$

The muon anomalous magnetic moment does not enter in Eq.(6). The correction of the muon anomalous magnetic moment (AMM) to the hyperfine splitting is conveniently represented by the use the experimental value $a_\mu = 1.16591981(62) \cdot 10^{-3}$ [1]:

$$\Delta E_{a_\mu}^{hfs}(nS) = a_\mu \Delta E_F^{hfs}(nS) = \begin{cases} 1S : & -1.598 \text{ meV} \\ 2S : & -0.200 \text{ meV} \end{cases}. \quad (7)$$

The contribution of the relativistic effects of order α^6 to the hyperfine structure is also known in the analytical form [3]:

$$\Delta E_{rel}^{hfs}(nS) = \left[1 + \frac{11n^2 + 9n - 11}{6n^2} (Z\alpha)^2 + \dots \right] \Delta E_F^{hfs}(nS) = \begin{cases} 1S : & -0.438 \text{ meV} \\ 2S : & -0.078 \text{ meV} \end{cases} \quad (8)$$

One-loop electron vacuum polarization correction in the interaction operator is determined by the following expression in the coordinate representation [22]:

$$\Delta V_{1\gamma,VP}^{hfs}(r) = \frac{8\alpha\mu_h}{3m_1m_2} \frac{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}{4} \frac{\alpha}{3\pi} \int_1^\infty \rho(s)ds \left[\pi\delta(\mathbf{r}) - \frac{m_e^2\xi^2}{r} \right] e^{-2m_e\xi r}, \quad (9)$$

where $\rho(\xi) = \sqrt{\xi^2 - 1}(2\xi^2 + 1)/\xi^4$. To obtain (9) it is necessary to use the following replacement in the photon propagator:

$$\frac{1}{k^2} \rightarrow \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi)d\xi \frac{1}{k^2 + 4m_e^2\xi^2}. \quad (10)$$

Averaging (9) over the wave functions (4) and (5), we find the correction of order α^5 to the hyperfine splitting:

$$\Delta E_{1\gamma,VP}^{hfs}(1S) = \frac{8\mu^3 Z^3 \alpha^5 \mu_h}{9m_1 m_p \pi} \int_1^\infty \rho(\xi)d\xi \left[1 - \frac{4m_e^2\xi^2}{W^2} \int_0^\infty x dx e^{-x(1+\frac{m_e\xi}{W})} \right] = -4.203 \text{ meV}, \quad (11)$$

$$\Delta E_{1\gamma,VP}^{hfs}(2S) = \frac{\mu^3 Z^3 \alpha^5 \mu_h}{9m_1 m_p \pi} \int_1^\infty \rho(\xi)d\xi \times \quad (12)$$

$$\left[1 - \frac{4m_e^2\xi^2}{W^2} \int_0^\infty x(1 - \frac{x}{2})^2 dx e^{-x(1+\frac{2m_e\xi}{W})} \right] = -0.540 \text{ meV}.$$

Changing the electron mass m_e to the muon mass m_1 in Eqs. (11), (12), we obtain the muon vacuum polarization contribution to the hyperfine splitting of order α^6 because the ratio $W/m_1 \ll 1$. The corresponding numerical values are included in Table 1. The contribution of the same order α^6 is given by the two-loop electron vacuum polarization diagrams in Fig.1(b,c,d). To derive the interaction operator corresponding the amplitude with two sequential loops in Fig.1(b) we use the double change (10). In the coordinate representation the interaction operator has the form:

$$\Delta V_{1\gamma,VP-VP}^{hfs}(r) = \frac{8\pi\alpha\mu_h}{3m_1m_p} \frac{\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2}{4} \left(\frac{\alpha}{3\pi} \right)^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \times \quad (13)$$

$$\times \left[\delta(\mathbf{r}) - \frac{m_e^2}{\pi r(\eta^2 - \xi^2)} \left(\eta^4 e^{-2m_e\eta r} - \xi^4 e^{-2m_e\xi r} \right) \right].$$

Corresponding correction to the hyperfine splitting of 1S and 2S levels can be presented as a three-dimensional integral over variables r , ξ and η . After that the integral over r is calculated analytically and over ξ , η numerically with the result:

$$\Delta E_{1\gamma,VP-VP}^{hfs}(1S) = \frac{8\alpha^6\mu^3 Z^3 \mu_h}{27m_1m_p} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \times \quad (14)$$

$$\times \left[1 - \frac{4m_e^2}{W^2(\eta^2 - \xi^2)} \int_0^\infty e^{-2x} x dx \left(\eta^4 e^{-2m_e\eta x/W} - \xi^4 e^{-2m_e\xi x/W} \right) \right] = -0.017 \text{ meV},$$

$$\Delta E_{1\gamma,VP-VP}^{hfs}(2S) = \frac{\alpha^6\mu^3 Z^3 \mu_h}{27m_1m_p} \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \times \quad (15)$$

$$\times \left[1 - \frac{4m_e^2}{W^2(\eta^2 - \xi^2)} \int_0^\infty e^{-x} x dx \left(\eta^4 e^{-2m_e \eta x/W} - \xi^4 e^{-2m_e \xi x/W} \right) \left(1 - \frac{x}{2} \right)^2 \right] = -0.002 \text{ meV}.$$

In a similar way we calculate the two-loop vacuum polarization contribution of order α^6 shown in Fig.1(c,d). In this case the potential is determined by the relation:

$$\Delta V_{1\gamma, 2-loop}^{hfs} V_P(r) = \frac{8\alpha^3 \mu_h}{3\pi m_1 m_p} \int_0^1 \frac{f(v) dv}{1-v^2} \left[\delta(\mathbf{r}) - \frac{m_e^2}{\pi r(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}} \right], \quad (16)$$

where the function

$$f(v) = v \left\{ (3-v^2)(1+v^2) \left[Li_2 \left(-\frac{1-v}{1+v} \right) + 2Li_2 \left(\frac{1-v}{1+v} \right) + \frac{3}{2} \ln \frac{1+v}{1-v} \ln \frac{1+v}{2} - \ln \frac{1+v}{1-v} \ln v \right] \right. \\ \left. + \left[\frac{11}{16} (3-v^2)(1+v^2) + \frac{v^4}{4} \right] \ln \frac{1+v}{1-v} + \left[\frac{3}{2} v(3-v^2) \ln \frac{1-v^2}{4} - 2v(3-v^2) \ln v \right] + \frac{3}{8} v(5-3v^2) \right\}, \quad (17)$$

$Li_2(z)$ is the Euler dilogarithm. Numerical contributions of the operator (16) to the HFS are included directly in Table 1. The role of the vacuum polarization effects in the HFS of the muonic helium ion extends further. There exists a number of contributions where the electron vacuum polarization enters the potential together with the nuclear structure, recoil and relativistic effects in the second order perturbation theory.

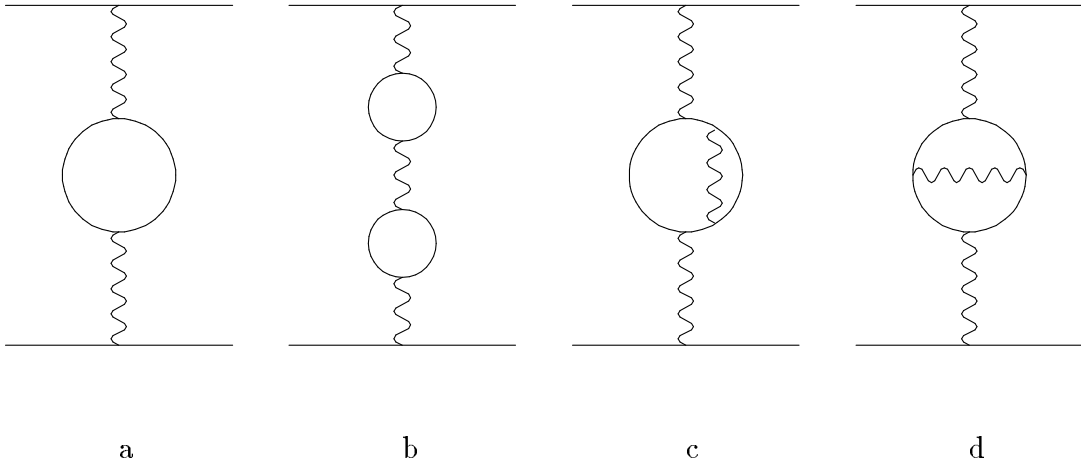


FIG. 1: Effects of one-loop and two-loop vacuum polarization in the one-photon interaction.

III. EFFECTS OF ONE-LOOP AND TWO-LOOP VACUUM POLARIZATION IN THE SECOND ORDER PERTURBATION THEORY

The second order perturbation theory (PT) corrections in the energy spectrum of hydrogen-like system are determined by the reduced Coulomb Green function \tilde{G} [23], whose partial expansion has the form:

$$\tilde{G}_n(\mathbf{r}, \mathbf{r}') = \sum_{l,m} \tilde{g}_{nl}(r, r') Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}'). \quad (18)$$

The radial function $\tilde{g}_{nl}(r, r')$ was presented in Ref.[23] in the form of the Sturm expansion in the Laguerre polynomials. The basic contribution of the electron vacuum polarization to HFS in the second order PT can be presented as follows (see Fig.2(a)):

$$\Delta E_{SOPT VP 1}^{hfs} = 2 \langle \psi | \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V_B^{hfs} | \psi \rangle, \quad (19)$$

where the modified Coulomb potential

$$\Delta V_{VP}^C(r) = \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \left(-\frac{Z\alpha}{r} \right) e^{-2m_e \xi r}. \quad (20)$$

The potential $\Delta V_B^{hfs}(r)$ is proportional to $\delta(\mathbf{r})$. So, we need the reduced Coulomb Green function with one zero argument. It was derived by means of the Hostler representation as a result of the subtraction of the pole term in Refs.[24, 25]:

$$\tilde{G}_{1S}(\mathbf{r}, 0) = \frac{Z\alpha\mu^2}{4\pi} \frac{e^{-x}}{x} g_{1S}(x), \quad g_{1S}(x) = [4x(\ln 2x + C) + 4x^2 - 10x - 2], \quad (21)$$

$$\tilde{G}_{2S}(\mathbf{r}, 0) = -\frac{Z\alpha\mu^2}{4\pi} \frac{e^{-x/2}}{2x} g_{2S}(x), \quad (22)$$

$$g_{2S}(x) = [4x(x-2)(\ln x + C) + x^3 - 13x^2 + 6x + 4],$$

where $C = 0.5772\dots$ is the Euler constant, $x = Wr$. Using Eqs.(21), (22) we can present necessary corrections in the HFS of the ion $(\mu_2^3 He)^+$ in the form:

$$\Delta E_{VP 1}^{hfs}(1S) = -\Delta E_F^{hfs}(1S) \frac{2\alpha}{3\pi} (1 + a_\mu) \int_1^\infty \rho(\xi) d\xi \int_0^\infty e^{-2x(1+\frac{m_e \xi}{W})} g_{1S}(x) dx = -9.260 \text{ meV}, \quad (23)$$

$$\Delta E_{VP 1}^{hfs}(2S) = \Delta E_F^{hfs}(2S) \frac{\alpha}{3\pi} (1 + a_\mu) \int_1^\infty \rho(\xi) d\xi \int_0^\infty e^{-x(1+\frac{2m_e \xi}{W})} g_{2S}(x) dx = -0.869 \text{ meV}. \quad (24)$$

The factor $(1 + a_\mu)$ is introduced in Eqs.(23), (24) so that these expressions contain corrections of orders α^5 and α^6 .

Two-loop contributions in Fig.2(b,c,d,e) are of order α^6 . Let us consider first contribution which is determined by the potentials (9), (20), the reduced Coulomb Green functions (21), (22) and the reduced Coulomb Green functions with both nonzero arguments. It is convenient to use the compact representation for it which was obtained in Refs.[24, 26]:

$$\tilde{G}_{1S}(r, r') = -\frac{Z\alpha\mu^2}{\pi} e^{-(x_1+x_2)} g_{1S}(x_1, x_2), \quad (25)$$

$$g_{1S}(x_1, x_2) = \frac{1}{2x_<} - \ln 2x_> - \ln 2x_< + Ei(2x_<) + \frac{7}{2} - 2C - (x_1 + x_2) + \frac{1 - e^{2x_<}}{2x_<},$$

$$\tilde{G}_{2S}(r, r') = -\frac{Z\alpha\mu^2}{16\pi x_1 x_2} e^{-(x_1+x_2)} g_{2S}(x_1, x_2), \quad (26)$$

$$\begin{aligned} g_{2S}(x_1, x_2) = & 8x_< - 4x_<^2 + 8x_> + 12x_<x_> - 26x_<^2x_> + 2x_<^3x_> - 4x_>^2 - 26x_<x_>^2 + 23x_<^2x_>^2 - \\ & - x_<^3x_>^2 + 2x_<x_>^3 - x_<^2x_>^3 + 4e^x(1 - x_<)(x_> - 2)x_> + 4(x_< - 2)x_<(x_> - 2)x_> \times \\ & \times [-2C + Ei(x_<) - \ln(x_<) - \ln(x_>)]. \end{aligned}$$

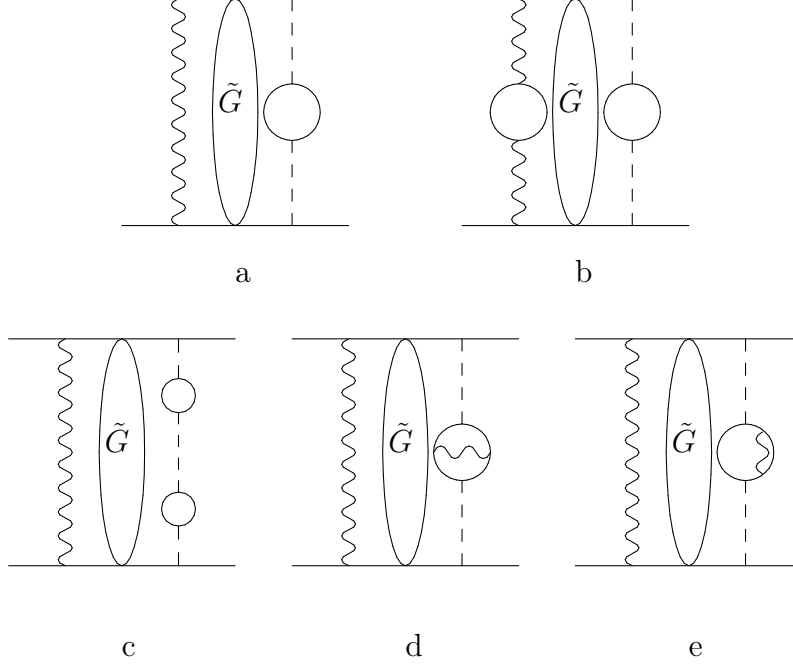


FIG. 2: Effects of one-loop and two-loop vacuum polarization in the second order perturbation theory (SOPT). The dashed line represents the Coulomb photon. \tilde{G} is the reduced Coulomb Green function.

Substituting Eqs.(9), (20), (25) and (26) in Eq.(19), we obtain two contributions for each energy level $1S$ and $2S$:

$$\Delta E_{SOPT \text{ VP } 21}^{hfs}(1S) = -\frac{16\alpha^6 Z^3 \mu^3 \mu_h (1 + a_\mu)}{27\pi^2 m_1 m_p} \int_1^\infty \rho(\xi) d\xi \times \quad (27)$$

$$\times \int_1^\infty \rho(\eta) d\eta \int_0^\infty dx e^{-2x(1+\frac{m_e \xi}{W})} g_{1S}(x),$$

$$\Delta E_{SOPT \text{ VP } 22}^{hfs}(1S) = -\frac{256\alpha^6 Z^3 \mu^3 \mu_h (1 + a_\mu) m_e^2}{27\pi^2 m_1 m_p W^2} \int_1^\infty \rho(\xi) d\xi \times \quad (28)$$

$$\times \int_1^\infty \rho(\eta) \eta^2 d\eta \int_0^\infty x_1 dx_1 e^{-2x_1(1+\frac{m_e \xi}{W})} \int_0^\infty x_2 dx_2 e^{-2x_2(1+\frac{m_e \xi}{W})} g_{1S}(x_1, x_2),$$

$$\Delta E_{SOPT \text{ VP } 21}^{hfs}(2S) = \frac{\alpha^6 Z^3 \mu^3 \mu_h (1 + a_\mu)}{27\pi^2 m_1 m_p} \int_1^\infty \rho(\xi) d\xi \times \quad (29)$$

$$\times \int_1^\infty \rho(\eta) d\eta \int_0^\infty \left(1 - \frac{x}{2}\right) dx e^{-x(1+\frac{2m_e \xi}{W})} g_{2S}(x),$$

$$\Delta E_{SOPT \text{ VP } 22}^{hfs}(2S) = -\frac{2\alpha^6 Z^3 \mu^3 \mu_h (1 + a_\mu) m_e^2}{27\pi^2 m_1 m_p W^2} \int_1^\infty \rho(\xi) d\xi \times \quad (30)$$

$$\times \int_1^\infty \rho(\eta) \eta^2 d\eta \int_0^\infty \left(1 - \frac{x_1}{2}\right) dx_1 e^{-x_1(1+\frac{2m_e \xi}{W})} \int_0^\infty \left(1 - \frac{x_2}{2}\right) dx_2 e^{-x_2(1+\frac{2m_e \xi}{W})} g_{2S}(x_1, x_2).$$

While contributions (27), (28) and (29), (30) are individually divergent, but their sum is finite. Corresponding numerical values are represented in Table 1. Corrections from two

other diagrams in the hyperfine structure can be calculated by the relations (23) and (24), in which the replacement of the potential (20) by the following potentials should be performed [21]:

$$\Delta V_{VP-VP}^C(r) = \left(\frac{\alpha}{3\pi}\right)^2 \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \left(-\frac{Z\alpha}{r}\right) \frac{1}{\xi^2 - \eta^2} \left(\xi^2 e^{-2m_e \xi r} - \eta^2 e^{-2m_e \eta r}\right), \quad (31)$$

$$\Delta V_{2-loop VP}^C(r) = -\frac{2Z\alpha^3}{3\pi^2 r} \int_0^1 \frac{f(v) dv}{(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}. \quad (32)$$

Omitting further intermediate expressions, which have the general structure quite similar to (23), (24), we include in Table 1 numerical values of corrections from potentials (31), (32).

IV. NUCLEAR STRUCTURE AND RECOIL EFFECTS

The main contribution of the nuclear structure to HFS of the S -energy levels including the Zemach correction, is determined by two-photon exchange diagrams (see Fig.3). We consider that the nuclear charge and magnetic moment are distributed in the space with definite densities. The vertex operator of the nucleus ${}^3_2\text{He}$ contains the electric G_E and magnetic G_M form factors which determine the interaction of the nucleus with the electromagnetic field. To calculate the nuclear structure correction of order α^5 we use the equation from Ref.[20] (the abbreviation "str" is used to designate the nuclear structure correction):

$$\begin{aligned} \Delta E_{str}^{hfs} = & -\frac{(Z\alpha)^5}{3\pi m_1 m_2 n^3} \delta_{l0} \int_0^\infty \frac{dk}{k} V(k), \quad (33) \\ V(k) = & \frac{2F_2^2 k^2}{m_1 m_2} + \frac{\mu}{(m_1 - m_2)k(k + \sqrt{4m_1^2 + k^2})} \left[-128F_1^2 m_1^2 - 128F_1 F_2 m_1^2 + 16F_1^2 k^2 + \right. \\ & + 64F_1 F_2 k^2 + 16F_2^2 k^2 + \frac{32F_2^2 m_1^2 k^2}{m_2^2} + \frac{4F_2^2 k^4}{m_1^2} - \frac{4F_2^2 k^4}{m_2^2} \left. \right] + \frac{\mu}{(m_1 - m_2)k(k + \sqrt{4m_2^2 + k^2})} \times \\ & \times \left[128F_1^2 m_2^2 + 128F_1 F_2 m_2^2 - 16F_1^2 k^2 - 64F_1 F_2 k^2 - 48F_2^2 k^2 \right]. \end{aligned}$$

To remove infrared divergence in Eq.(33) we must take into account the contribution of the iterative term of the quasipotential to the HFS of the atom $(\mu^3\text{He})^+$:

$$\Delta E_{iter,str}^{hfs} = - \langle V_{1\gamma} \times G^f \times V_{1\gamma} \rangle_{str}^{hfs} = -\frac{64}{3} \frac{\mu^4 Z^4 \alpha^5 \mu_h}{m_1 m_p \pi n^3} \int_0^\infty \frac{dk}{k^2}, \quad (34)$$

where the angle brackets denote the averaging of the interaction operator over the Coulomb wave functions and the index "hfs" is indicative of the hyperfine part in the iterative term of the quasipotential. $V_{1\gamma}$ is the quasipotential of the one-photon interaction, G^f is the free two-particle propagator. The integration in Eqs.(33) and (34) can be done by means of the dipole parameterization for the Pauli form factor F_1 and the Dirac form factor F_2 [27, 28]. The parameter for this parameterization Λ^2 can be related with the nuclear charge radius r_N : $\Lambda^2 = 12/r_N^2$. Numerical value of $r_N = 1.844 \pm 0.045$ fm is taken from Ref.[8]. Since

the dependence on the principal quantum number n in Eq.(33) is determined by the factor $1/n^3$, the numerical values of the nuclear structure corrections for the states $1S$ and $2S$

$$\Delta E_{str}^{hfs} = \begin{cases} 1S : 48.376 \text{ meV} \\ 2S : 6.047 \text{ meV} \end{cases} \quad (35)$$

are cancelled in the special hyperfine splitting interval $\Delta_{12} = (8\Delta E_{str}^{hfs}(2S) - \Delta E_{str}^{hfs}(1S))$. So, the calculation of the interval Δ_{12} is free of the uncertainty connected with the nuclear structure at least in the leading order. The value of the correction (33) is dependent on the form of distributions $G_{E,M}$. The replacement of the dipole parameterization by the Gaussian model leads to the change of the numerical value (35) by 2 per cent.

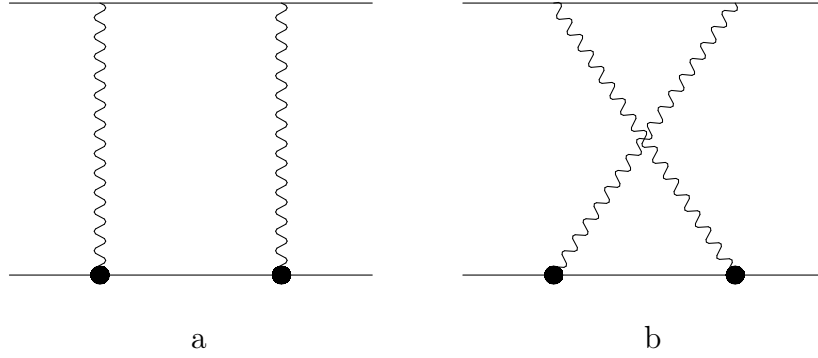


FIG. 3: Effects of the nuclear structure of order α^5 .

The sixth order over α contribution to the HFS shown in Fig.4 contains both the nuclear structure and the vacuum polarization effects. Using the substitution (10) and Eq.(33) it can be written as follows:

$$\Delta E_{str,VP}^{hfs} = -\frac{2\alpha(Z\alpha)^5\mu^3}{m_1m_2\pi^2n^3} \int_0^\infty V_{VP}(k)dk \int_0^1 \frac{v^2 \left(1 - \frac{v^2}{3}\right) dv}{k^2(1-v^2) + 4m_e^2}, \quad (36)$$

where the potential $V_{VP}(k)$ differs on the expression $V(k)$ in Eq.(33) by the additional factor k^2 . The amplitude contribution (36) to the energy spectrum should be augmented by two iterative terms which are presented in Fig.4(c,d):

$$\begin{aligned} \Delta E_{iter,str VP}^{hfs} &= -2 \langle V^C \times G^f \times \Delta V_{VP}^{hfs} \rangle^{hfs} = -2 \langle V_{VP}^C \times G^f \times \Delta V_B^{hfs} \rangle^{hfs} = \\ &= -\Delta E_F^{hfs} \frac{4\mu\alpha(Z\alpha)}{m_e\pi^2} \int_0^\infty dk \int_0^1 \frac{v^2 \left(1 - \frac{v^2}{3}\right) dv}{k^2(1-v^2) + 1}. \end{aligned} \quad (37)$$

Numerically the sum of corrections (36) and (37) is equal

$$\Delta E_{str,VP}^{hfs} + 2\Delta E_{iter,str VP}^{hfs} = \begin{cases} 1S : 0.760 \text{ meV} \\ 2S : 0.095 \text{ meV} \end{cases} \quad (38)$$

There exists another contribution of the nuclear structure in the second order PT, which is determined by the hyperfine term of the Breit Hamiltonian and the operator of the one-photon interaction (see Fig.5(b))

$$\Delta V_{str} = \frac{2\pi(Z\alpha)}{3} r_N^2 \delta(\mathbf{r}). \quad (39)$$

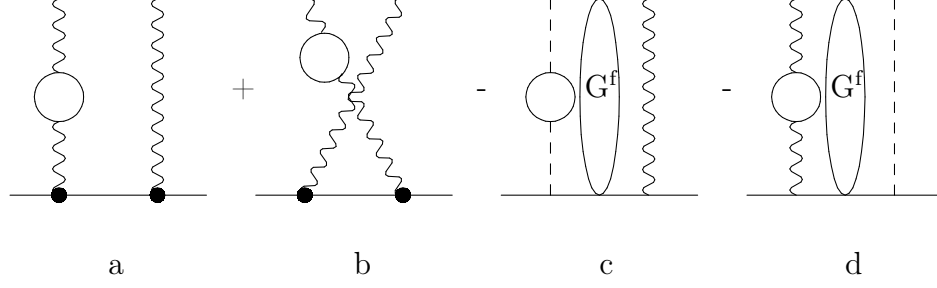


FIG. 4: Nuclear structure and vacuum polarization effects of order α^6 . The dashed line represents the Coulomb photon. G^f is the free two-particle propagator.

The nuclear structure effects are taken into account in Eq.(39) in terms of the nuclear charge radius r_N . This contribution has the form:

$$\Delta E_{str}^{hfs}{}_{SOPT}(nS) = 2 \langle \psi_n | \Delta V_B^{hfs} \cdot \tilde{G} \cdot \Delta V_{str} | \psi_n \rangle = \frac{4\pi(Z\alpha)}{3} \Delta E_F^{hfs}(nS) r_N^2 \tilde{G}(0,0). \quad (40)$$

The value of the reduced Coulomb Green function at zero arguments in the coordinate representation $\tilde{G}(0,0)$ is divergent. The reason of the appeared divergence lies in the expansion of the potentials in Eq.(40) at small relative momenta and further integration (40) at all values of relative momenta. To calculate the quantity $\tilde{G}(0,0)$ the dimensional regularization can be useful [29, 30, 31]. Subtracting the iterative term $2 \langle \psi_n | \Delta V_B^{hfs} \cdot G^f \Delta V_{str} | \psi_n \rangle$ from Eq.(40) we obtain the following result:

$$\Delta E_{str}^{hfs}{}_{SOPT}(1S) = \frac{4}{3} (Z\alpha)^2 m_1^2 r_N^2 \Delta E_F^{hfs}(1S) \left[\ln(Z\alpha) - \frac{3}{2} \right], \quad (41)$$

$$\Delta E_{str}^{hfs}{}_{SOPT}(2S) = \frac{4}{3} (Z\alpha)^2 m_1^2 r_N^2 \Delta E_F^{hfs}(2S) [\ln(Z\alpha) - \ln 2]. \quad (42)$$

One further contribution of the sixth order over α can be derived from the one-photon amplitude (Fig.5(a)) expanding the nuclear magnetic formfactor at small values of the relative momenta. As a result the potential of the hyperfine interaction (3) in the coordinate representation gains the additional term

$$\Delta V_{1\gamma str}^{hfs}(r) = -\frac{4\pi\alpha(1+a_\mu)}{9m_1m_p} r_M^2 \frac{\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2}{4} \nabla^2 \delta(\mathbf{r}), \quad (43)$$

where r_M is the nuclear magnetic radius. For the averaging the operator (43) over the Coulomb wave functions we use the following relation

$$\int \nabla^2 \delta(\mathbf{r}) d\mathbf{r} |\psi_n(\mathbf{r})|^2 = 2 \left(\psi(0) \nabla^2 \psi(0) + \left(\frac{d\psi_n}{dr} \right)^2 \Big|_{r=0} \right), \quad (44)$$

and the value $\nabla^2 \psi(0) = \psi(0) \mu^2 (Z\alpha)^2 \frac{3+2(n^2-1)}{n^2}$ [17, 32]. As a result we obtain the important correction proportional to the nuclear magnetic radius:

$$\Delta E_{1\gamma str}^{hfs}(nS) = -\frac{4}{3} (Z\alpha)^2 \mu^2 r_M^2 \Delta E_F^{hfs}(nS) \frac{1-n^2}{4n^2}. \quad (45)$$

TABLE I: Hyperfine structure of $1S$ and $2S$ states in the ion of muonic helium $(\mu \text{ } ^3_2\text{He})^+$. $\Delta_{12} = (8\Delta E^{hfs}(2S) - \Delta E^{hfs}(1S))$.

Contribution to HFS	$1S$, meV	$2S$, meV	Interval Δ_{12}	Reference
The Fermi energy	-1370.725	-171.341	0	(6), [3, 4]
Muon AMM correction of orders α^5 and α^6	-1.598	-0.200	0	(7), [3, 4]
Relativistic correction of order α^6	-0.438	-0.078	-0.183	(8), [3]
One-loop VP contribution in 1γ interaction of order α^5	-4.203	-0.540	-0.119	(11)-(12)
Two-loop VP contribution in 1γ interaction of order α^6	-0.050	-0.004	0.016	(14)-(16)
One-loop muon VP contribution in 1γ interaction of order α^6	-0.052	-0.007	0	(11)-(12)
One-loop VP contribution in the second order PT of order α^5	-9.260	-0.869	2.305	(23)-(24)
Two-loop VP contribution in the second order PT of order α^6	-0.105	-0.010	0.022	(27)-(30)
Nuclear structure correction of order α^5	48.376	6.047	0	(33),(34)
Contribution of VP and nuclear structure of order α^6	0.760	0.095	0	(38)
Nuclear structure correction of order α^6	2.553	0.272	-0.377	(41)-(42),(45)
Nuclear structure and muon self-energy correction of order α^6	-0.145	-0.018	0	(46), [31]
Nuclear recoil correction of order α^6	0.330	0.038	-0.026	(47)-(48),[34]
Summary contribution	-1334.560	-166.615	1.638	

We have included in Table 1 the total nuclear structure contribution which is determined by expressions (41), (42) and (45) for $1S$ and $2S$ energy levels at $r_M \approx r_N$. Let us write here also the corection of order α^6 connected with the nuclear structure and the muon self energy [31]:

$$\Delta E_{str \ SE}^{hfs} = \frac{5}{2} \frac{\alpha(Z\alpha)}{\pi} m_1 R_Z \Delta E_F^{hfs} = \begin{cases} 1S : & -0.145 \text{ meV} \\ 2S : & -0.018 \text{ meV} \end{cases} \quad (46)$$

where R_Z is the Zemach radius.

One part of the recoil corrections to the HFS is accounted in the calculation of the diagrams in Figs.3-4. Thus the leading order recoil contribution $(Z\alpha)(m_1/m_2) \ln(m_1/m_2) \Delta E_F^{hfs}$ is contained in the potential (34). The recoil correction of order $(Z\alpha)^2(m_1/m_2) \Delta E_F^{hfs}$ for the ground state HFS in the hydrogen atom was calculated in Refs.[31, 33] and for the hyperfine interval Δ_{12} in Ref.[34]. Using these results we can present the analytical expressions for the

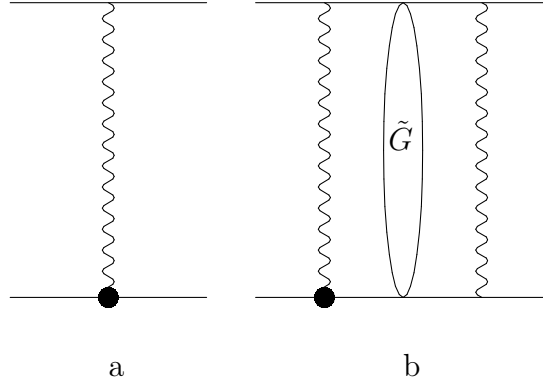


FIG. 5: Nuclear structure effects of order α^6 in the one-photon interaction and the second order PT. \tilde{G} is the reduced Coulomb Green function.

recoil corrections and their numerical values for the HFS of $1S$ and $2S$ -states in the form:

$$\begin{aligned} \Delta E_{rec}^{hfs}(1S) = (Z\alpha)^2 \frac{\mu^2}{m_1 m_2} \Delta E_F^{hfs}(1S) & \left[-\frac{17}{12} + \frac{25}{3\zeta} + \frac{31\zeta}{72} + \right. \\ & \left. + \ln 2 \left(\frac{1}{2} - \frac{23}{2\zeta} - \frac{11\zeta}{8} \right) + \ln \left(\frac{1}{Z\alpha} \right) \left(-\frac{3}{2} + \frac{7}{2\zeta} + \frac{7\zeta}{8} \right) \right] = 0.330 \text{ meV}, \end{aligned} \quad (47)$$

$$\begin{aligned} \Delta E_{rec}^{hfs}(2S) = (Z\alpha)^2 \frac{\mu^2}{m_1 m_2} \Delta E_F^{hfs}(2S) & \left[-\frac{265}{96} + \frac{821}{96\zeta} - \frac{809\zeta}{1152} + \right. \\ & \left. + \ln 2 \left(1 - \frac{12}{\zeta} - \frac{\zeta}{2} \right) + \ln \left(\frac{1}{Z\alpha} \right) \left(-\frac{3}{2} + \frac{7}{2\zeta} + \frac{7\zeta}{8} \right) \right] = 0.038 \text{ meV}, \end{aligned} \quad (48)$$

where $\zeta = 2m_2\mu_h/m_p Z$.

V. SUMMARY AND CONCLUSION

In this work various QED corrections, effects of the nuclear structure and recoil of orders α^5 and α^6 have been calculated for the hyperfine splittings of $1S$ and $2S$ energy levels in the ion of muonic helium $(\mu_2^3\text{He})^+$. The investigation of the energy structure of $1S$ and $2S$ states in this atom has the clear experimental prospect. Contrary to earlier performed studies of the energy spectra of light muonic atoms in Refs.[6, 7, 8] we use the three-dimensional quasipotential method for the description of the muon and helion bound state. All corrections considered here can be separated into two groups. The first group consists of the contributions which are specific for muonic helium ion. Primarily they are connected with the effects of the electron vacuum polarization. In our study these contributions are presented in the integral form and obtained numerically. The corrections known in the analytical form from the calculations of the hyperfine structure of muonium and hydrogen atom enter in the second group [3]. Numerical values of all corrections are presented in Table 1. It contains also several basic references on the papers where the precision calculations of the HFS of simple atoms were considered. Other references can be founded in the review articles [3, 4].

As mentioned above, the hyperfine structure of light exotic atoms was investigated on the basis of the Dirac equation many years ago in Refs.[10, 11]. The energies of the transitions ($2S-2P$) were obtained for the muonic hydrogen and ion of muonic helium (μ_2^3He)⁺. In this calculation only basic contributions to the HFS with the precision 0.1 meV were accounted. It follows from the Table 2 of Ref.[11], that the energies of the transitions ($^1S_{1/2}-^3P_{1/2}$) and ($^3S_{1/2}-^3P_{1/2}$) are equal correspondingly 1167.3 meV and 1334.1 meV resulting the definite value of the hyperfine splitting of 2S-state: -166.8 meV. The total value of the HFS of 2S energy level entering in our Table 1 -166.615 meV is in good agreement with this result. So, the calculation of the HFS in the ion of muonic helium performed in this work improves the obtained earlier result in [11] for the 2S state by the calculation of order α^6 corrections and gives new result for the hyperfine splitting of 1S state. The estimate of the next to considered order contribution over α has the form: $\alpha^3 \ln(1/\alpha) \Delta E_F^{hfs}(1S) \approx 0.003$ meV. Despite the fact that all contributions in Table 1 are written with the accuracy 0.001 meV, the precision of our calculation of the HFS is not so high. The reason is that the nuclear structure correction of order α^5 has the largest theoretical uncertainty associated with the errors in the measurement of electromagnetic form factors for the nucleus 3_2He . When the dipole parameterization for the form factors is used the value of the theoretical uncertainty is determined by the error of the nuclear charge radius: $r_N(^3_2He) = 1.844 \pm 0.045$ fm. As a result the theoretical error can reach near ± 1.5 meV for 1S-level and ± 0.20 meV for 2S-level. Nuclear corrections to the HFS connected with the motion of the nucleons forming the nucleus 2H , 3H , 3He of light hydrogen-like atoms were studied in Refs.[35, 36]. Another source of the uncertainty is connected with the nuclear polarizability effect [37, 38, 39, 40, 41, 42, 43, 44, 45]. It demands further investigation on the basis of the experimental data on the polarized lepton scattering by the nucleus 3_2He . The value of the nuclear polarizability contribution to the HFS of the muonic helium ion can amount to several meV. The nuclear polarizability contribution should be considered in the combination with the nuclear corrections because both effects are associated with the interaction of the multinucleon system with electromagnetic field. The hyperfine splitting interval Δ_{12} has not uncertainties conditioned by the nuclear structure and polarizability corrections. So, the obtained value of the interval $\Delta_{12} = 1.638$ meV can be used for the check of QED predictions in the case of muonic helium ion with the accuracy 0.01 meV.

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